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ON THE SMOOTHNESS OF SOLUTIONS TO A SPECIAL NEUMANN PROBLEM ON NONSMOOTH DOMAINS

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Abstract

In this paper, we show that the weak solution to the Neumann problem for the differential operator $-\Delta u + \lambda u$ in a domain whose boundary is a curvilinear polygon, is not merely in H^1 but in H^s, where $s \in (3/2, 2]$ depends on the interior angles at the corners. We apply this result to the reconstruction problem of turbulent layers in adaptive optics.

1. Introduction

The problem we are dealing with in this paper is motivated by a problem from adaptive optics.

Since the wavefront of incoming light is distorted by atmospheric turbulences, modern ground based telescopes use adaptive optics systems, where one or more deformable mirrors are used to correct for phase perturbations (see, e.g., [7]). In the case of multi conjugate

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adaptive optics (MCAO), multiple wavefront sensors as well as multiple deformable mirrors are employed to achieve a high degree of wavefront correction. The wavefront measurements from multiple directions are used to solve the so-called atmospheric tomography problem, i.e., the reconstruction of turbulent layers above the telescope. This problem is severly ill-posed and has to be solved via regularization methods (see, e.g., [1]). In [2, 6], this problem was treated with a Landweber-Kaczmarz regularization method.

In this iteration method, the final step for the computation of the H^1-iterates consists in applying the adjoint of the embedding operator from $H^1(\Omega_l)$ into $L^2(\Omega_l)$ for each turbulent layer $l = 1, \dots, L$, where

$$\Omega_l := \bigcup_{k=1}^{G} \{(x, y) \in \mathbb{R}^2 : r_i^2 \le (x - h_l a_k)^2 + (y - h_l b_k)^2 \le r_o^2\},$$

$(a_k, b_k) \in \mathbb{R}^2$, $h_l > 0$, and G is the number of guide stars, i.e., Ω_l is a union of annuli with inner radius r_i and outer radius r_o (see Figure 1.1).

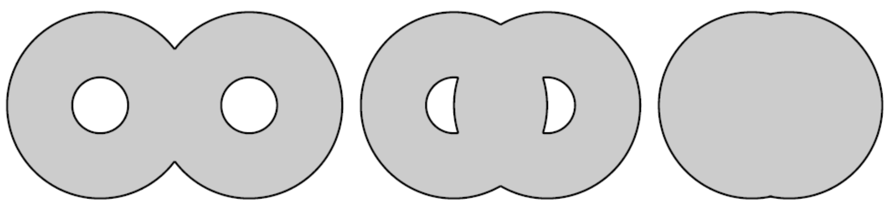

Figure 1.1. Three different regions $\Omega_l : r_o = 10$, $r_i = 3$, $G = 2$, $(a_2, b_2) - (a_1, b_1) = (z, 0)$ with $z = 16$ (left), $z = 10$ (middle), and $z = 4$ (right).

To have some freedom, the authors of [2] use the following weighted inner product in H^1:

$$\langle \nabla u, \nabla v \rangle_{L^2} + \lambda \langle u, v \rangle_{L^2}, \qquad \lambda > 0.$$

Let $i : H^1(\Omega) \to L^2(\Omega)$ denote the embedding operator and let $f \in L^2(\Omega)$, then $u := i^* f$ is the solution to the problem

$$\langle \nabla u, \nabla v \rangle_{L^2(\Omega)} + \lambda \langle u, v \rangle_{L^2(\Omega)} = \langle f, v \rangle_{L^2(\Omega)},$$

for all $v \in H^1(\Omega)$, or equivalently, $u \in H^1(\Omega)$ is the weak (variational) solution of the boundary value problem

$$-\Delta u + \lambda u = f \quad \text{in } \Omega,$$

$$\frac{\partial u}{\partial \nu} = 0 \quad \text{on } \partial\Omega. \tag{1.1}$$

Considerations in adaptive optics show that the turbulent layers are usually smoother than H^1, namely, they are in H^s with $s \approx 11/6$ (cf. [2, Subsection 3.1]). Therefore, it would be advantageous if the regularized solutions were also smoother than H^1. It turns out that this is actually the case.

It is well-known that the solution u to (1.1) even satisfies $u \in H^2(\Omega)$, if the boundary Γ is $C^{1,1}$ or if Ω is convex (see, e.g., [3, Sections 2 and 3]). It was shown in [3, Section 4] that the solution is an element of $H^s(\Omega)$ with $s \in (3/2, 2)$ if Ω is a rectilinear polygon with non-convex corners, where the value s depends on the angles larger than π. This result was used in [2], where it was applied to annular regions where the boundary is approximated by rectilinear polygons.

In the next section, we prove similar results for curvilinear polygons. We follow the ideas of [3, Subsection 5.2], where such results have been shown for Dirichlet problems. It turns out that the study of Neumann problems is more difficult and not just a straight forward extension.

In the last section, some useful results about Sobolev spaces and standard results about the solution to elliptic boundary value problems from [3] are collected for the convenience of the readers.

2. Smoothness of Solutions to Neumann Problems

We now fix some notations concerning the sets Ω, we want to deal with (compare [3, Subsection 1.5.2]): Ω is an open bounded connected subset of $\mathbb{R}^2$, whose boundary is a curvilinear polygon of class $C^{1,1}$ (see [3, Definition 1.4.5.1]). The interior of each of the $C^{1,1}$ curves of the boundary is denoted by Γ_j with $j = 1, \ldots, N$. The curve Γ_{j+1} follows Γ_j according to the positive orientation on each connected component of Γ, i.e., counter-clockwise at the outer boundary and clockwise for interior curves. The endpoints of $\overline{\Gamma_j}$, i.e., the corners, are denoted by S_j.

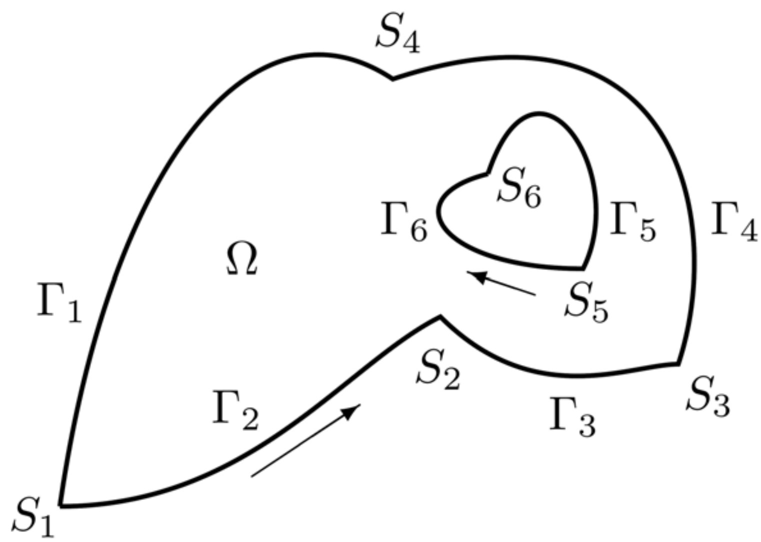

Figure 2.1. Curvilinear polygon.

Thus, usually, S_j will be the starting point of the curve $\overline{\Gamma_{j+1}}$, except for the endpoints of the outer and inner boundaries. Figure 2.1 shows a curvilinear polygon with one interior curve: The outer boundary has 4 corners and the interior one 2; S_4 is the starting point of $\overline{\Gamma_1}$ instead of $\overline{\Gamma_5}$ and S_6 is the starting point of $\overline{\Gamma_5}$ instead of $\overline{\Gamma_7}$.

Moreover, let $x_j(\sigma)$ be the point on Γ, whose distance to S_j is σ. Thus, for $|\sigma|$ small enough, $x_j(\sigma) \in \Gamma_j$ for $\sigma < 0$ and $x_j(\sigma) \in \Gamma_{j+1}$ for $\sigma > 0$. The interior angle between the tangents of the boundary curves at the corner S_j, i.e., between $x_j'(0^+)$ and $x_j'(0^-)$, is denoted by $\omega_j \in (0, 2\pi)\setminus\{\pi\}$.

Finally, ν_j denotes a $C^{0,1}$ vector field in a neighbourhood of $\overline{\Omega}$, which is the unit outward normal a.e. on Γ_j.

Allowing non homogeneous boundary conditions, Equation (1.1) turns into

$$\begin{aligned} Lu &:= -\Delta u + \lambda u = f \quad \text{in } \Omega, \\ \frac{\partial u}{\partial \nu_j} &= g_j \quad \text{on } \partial\Gamma_j, \quad j = 1, \dots, N, \end{aligned} \tag{2.1}$$

where $f \in L^2(\Omega)$, $g_j \in H^{\frac{1}{2}}(\Gamma_j)$, and $\lambda > 0$. The variational formulation for this problem reads as

$$\langle \nabla u, \nabla v \rangle_{L^2(\Omega)} + \lambda \langle u, v \rangle_{L^2(\Omega)} = \langle f, v \rangle_{L^2(\Omega)} + \sum_{j=1}^{N} \langle g_j, v \rangle_{L^2(\Gamma_j)}, \tag{2.2}$$

for all $v \in H^1(\Omega)$. In the latter inner product instead of v one should actually write $\gamma_j v$, where γ_j is the trace operator mapping a function onto its restriction on Γ_j. Wherever it is clear from the context we omit γ_j. Noting that $\partial\Omega$ is of class $C^{0,1}$, the existence and uniqueness of the solution $u \in H^1(\Omega)$ of (2.2) follows with the Lax-Milgram lemma and Theorem 3.5. Moreover, we obtain the estimate

$$\|u\|_{H^1(\Omega)} \leq \lambda^{-1}(1 + \lambda)^{\frac{1}{2}} \left(\|f\|_{L^2(\Omega)} + c\lambda^{\frac{1}{2}} \sum_{j=1}^{N} \|g_j\|_{L^2(\Gamma_j)} \right), \tag{2.3}$$

for some $c > 0$. This estimate shows that to guarantee the existence of a solution $u \in H^1(\Omega)$, the functions g_j not necessarily have to be elements of $H^{\frac{1}{2}}(\Gamma_j)$. However, if we want to prove higher smoothness of the solution u, i.e., $u \in H^s(\Omega)$ with $s > 1$, then $g_j \in H^{\frac{1}{2}}(\Gamma_j)$ is needed.

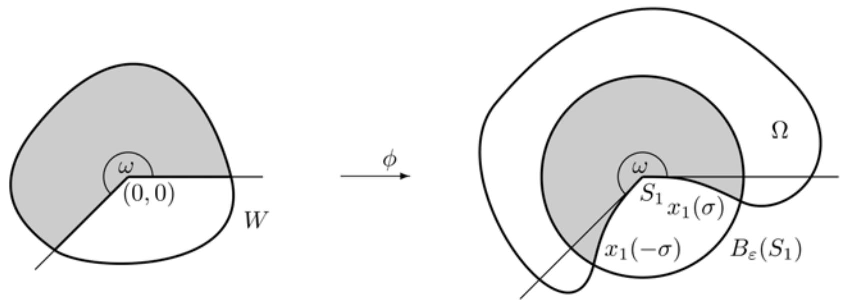

Figure 2.2. Transform of corner region.

The proof of the smoothness of the solution u is based on results for regions with $C^{1,1}$ boundaries and for rectilinear polygons (see [3, Sections 2 and 4] and Theorems 3.9 and 3.10 below) combined with a perturbation theory for problem (2.1) after a local transform of the region close to a corner onto a set, where the boundary is rectilinear near the corner with the angle unchanged. The transform we need is described in the following remark:

Remark 2.1. Let us assume that Ω is even simply connected with only one corner $(N = 1)$ and that (after a rotation)

$$x_1'(0^+) = (1, 0) \quad \text{and} \quad x_1'(0^-) = -(\cos\omega_1, \sin\omega_1).$$

For $|\sigma|$ sufficiently small, we define the $C^{1,1}$ functions

$$\kappa_1(\sigma) := \begin{cases} x_1(-\sigma), & \sigma \geq 0, \\ S_1 + \sigma(\cos\omega_1, \sin\omega_1), & \sigma < 0, \end{cases} \qquad \kappa_2(\sigma) := \begin{cases} x_1(\sigma), & \sigma \geq 0, \\ S_1 + \sigma(1, 0), & \sigma < 0. \end{cases}$$

Moreover, we define the transform

$$\phi(x, y) := \kappa_2\left(x - y\frac{\cos\omega_1}{\sin\omega_1}\right) + \kappa_1\left(\frac{y}{\sin\omega_1}\right) - S_1.$$

Due to the inverse function theorem, a neighbourhood W of $(0, 0)$ and an $\varepsilon > 0$ exist such that (see Figure 2.2 for a possible transform of regions):

(i) $\phi : \overline{W} \to \overline{B_\varepsilon(S_1)}$ is bijective, ϕ and its inverse $\psi = \phi^{-1}$ are $C^{1,1}$ transforms.

(ii) $D\phi(0, 0) = I = D\psi(S_1)$ and $0 < c_1 < \det D\phi(x, y) < c_2 < \infty$ for all $(x, y) \in \overline{W}$ and some constants c_1, c_2. Here $D\phi$ denotes the Jacobian of ϕ.

(iii) $\phi(\sigma, 0) = x_1(\sigma)$ for all $(\sigma, 0) \in \overline{W}$, $\sigma > 0$,

$\phi(\sigma\cos\omega_1, \sigma\sin\omega_1) = x_1(-\sigma)$ for all $(\sigma\cos\omega_1, \sigma\sin\omega_1) \in \overline{W}$, $\sigma > 0$.

(iv) $\phi(W \cap \{(\sigma\cos\theta, \sigma\sin\theta) : \sigma > 0, 0 < \theta < \omega_1\}) = B_\varepsilon(S_1) \cap \Omega$.

Remark 2.2. Under a transform ϕ as in the remark above, also a differential operator and boundary conditions are transformed: Let $\widetilde{\Omega} \subset B_\varepsilon(S_1) \cap \Omega$ be an open domain, whose boundary is a curvilinear polygon of class $C^{1,1}$ with only one corner S_1 that coincides with the boundary of Ω near S_1. Thus, $\widetilde{\Gamma}_1 := \partial\widetilde{\Omega} \setminus S_1$ is a $C^{1,1}$ curve.

If u solves the boundary value problem

$$Lu := -\Delta u + \lambda u = f \quad \text{in } \widetilde{\Omega},$$

$$\frac{\partial u}{\partial \nu_1} = g_1 \quad \text{on} \quad \widetilde{\Gamma}_1,$$

with $f \in L^2(\tilde{\Omega})$ and $g_1 \in H^{\frac{1}{2}}(\tilde{\Gamma}_1)$, then it is an immediate consequence of the chain rule and conditions (i)-(iv) in Remark 2.1 that $w := u \circ \phi$ solves the boundary value problem

$$Aw = (Lu) \circ \phi = f \circ \phi \quad \text{in } \Omega' := \psi(\tilde{\Omega}),$$

$$\frac{\partial w}{\partial \nu_{A,1}} = -(g_1 \circ \phi)\det D\phi \quad \text{on } \Gamma_1' := \partial\Omega' \setminus (0,\, 0),$$

where A is again a strongly elliptic differential operator of the form

$$Aw := \sum_{i,\, j=1}^{2} D_i(a_{ij} D_j w) + \sum_{i=1}^{2} a_i D_i w + \lambda w. \tag{2.4}$$

The coefficients of A satisfy the conditions

$$a_{ij} \in C^{0,1}(\overline{\Omega}'), \quad a_i \in L^\infty(\Omega'), \quad \lambda > 0,$$

$$a_{12} = a_{21}, \quad a_{11}(0,\, 0) = -1 = a_{22}(0,\, 0), \quad a_{12}(0,\, 0) = 0,$$

$$\sum_{i,\, j=1}^{2} a_{ij}(x,\, y)\xi_i \xi_j \le -\alpha(\xi_1^2 + \xi_2^2), \tag{2.5}$$

for all $(x,\, y) \in \overline{\Omega}'$, $(\xi_1,\, \xi_2) \in \mathbb{R}^2$, and some $\alpha > 0$.

The expression $\dfrac{\partial w}{\partial \nu_{A,1}}$ denotes the conormal derivative defined by

$$\frac{\partial w}{\partial \nu_{A,1}} = \sum_{i,\, j=1}^{2} a_{ij} \nu_1^i D_j w,$$

where ν_1^i is the i-th component of the unit outward normal at the boundary Γ_1'.

Note that Ω' satisfies the conditions:

(i) Ω' is an open bounded subset of $\mathbb{R}^2$, whose boundary is a curvilinear polygon of class $C^{1,1}$ with only one corner at $(0, 0)$. Note that $\partial\Omega'$ is of class $C^{0,1}$.

(ii) The boundary $\Gamma_1' := \partial\Omega' \setminus (0, 0)$ is rectilinear close to $(0, 0)$ with interior angle $\omega_1 \in (0, 2\pi)\setminus\{\pi\}$.

As in [3, Subsection 5.2], we need some preparatory lemmata for regions with only one corner.

Lemma 2.3. *Let* $\Omega' \subset \mathbb{R}^2$ *satisfy the conditions* (i) *and* (ii) *in Remark* 2.2 *and let* A *be a differential operator of the form* (2.4)*, where the coefficients satisfy the conditions in* (2.5)*. Then, there exists a constant* $c > 0$ *such that*

$$\| w \|_{H^2(\Omega')} \leq c \left(\| Aw \|_{L^2(\Omega')} + \left\| \frac{\partial w}{\partial \nu_{A,1}} \right\|_{H^{\frac{1}{2}}(\Gamma_1')} + \| w \|_{H^1(\Omega')} \right), \tag{2.6}$$

for all $w \in H^2(\Omega')$*. Moreover, it holds that the operator* $T_A : H^2(\Omega') \to L^2(\Omega') \times H^{\frac{1}{2}}(\Gamma_1')$*, defined by*

$$T_A w := \left\{ Aw, \frac{\partial w}{\partial \nu_{A,1}} \right\}, \tag{2.7}$$

has a finite-dimensional null-space and closed range, i.e., it is a semi-Fredholm operator.

Proof. Let $O \subset \Omega'$ be an open domain, whose boundary is a rectilinear polygon that coincides with $\partial\Omega'$ close to $(0, 0)$. It is always possible to choose O such that all other angles (except ω_1) are in the interval $(0, \pi)$. Moreover, let $w \in H^2(\Omega')$ and let η be a C^∞ cut-off function equal to 1 near $(0, 0)$ and with support in O. It will turn out later on how small the support should be.

In a first step, we consider ηw : estimate (3.4) implies that

$$\|\eta w\|_{H^2(\Omega')} \leq c_1\left(\|A(\eta w)\|_{L^2(\Omega')} + \left\|\frac{\partial(\eta w)}{\partial\nu_{A,1}}\right\|_{H^{\frac{1}{2}}(\Gamma_1')}\right.$$

$$\left. + \|(A+\Delta)(\eta w)\|_{L^2(\Omega')} + \left\|\frac{\partial(\eta w)}{\partial\nu_{A,1}} + \frac{\partial(\eta w)}{\partial\nu}\right\|_{H^{\frac{1}{2}}(\Gamma_1')}\right), \tag{2.8}$$

for some $c_1 > 0$. Obviously,

$$\|(A+\Delta)(\eta w)\|_{L^2(\Omega')} \leq 2c(\eta)\|\eta w\|_{H^2(\Omega')} + c_2\|\eta w\|_{H^1(\Omega')}, \tag{2.9}$$

for some $c_2 > 0$ and

$$c(\eta) := \max_{(x,y)\in\operatorname{supp}(\eta)\cap\overline{\Omega}'}\{|a_{11}(x, y)+1|, |a_{22}(x, y)+1|, |a_{12}(x, y)|\}.$$

Noting that Γ_1' is of class $C^{1,1}$ and that

$$\|g\|^2_{H^{\frac{1}{2}}(\Gamma_1')} = \|g\|^2_{L^2(\Gamma_1')} + \int_{\Gamma_1'}\int_{\Gamma_1'}\frac{|g(\xi)-g(\tau)|^2}{|\xi-\tau|^2}\,d\sigma(\xi)d\sigma(\tau),$$

we obtain together with

$$\frac{\partial(\eta w)}{\partial\nu_{A,1}} + \frac{\partial(\eta w)}{\partial\nu_1} = ((a_{11}+1)\nu_1^1 + a_{12}\nu_1^2)(\eta w)_x$$

$$+ (a_{12}\nu_1^1 + (a_{22}+1)\nu_1^2)(\eta w)_y,$$

the estimate

$$\left\|\frac{\partial(\eta w)}{\partial\nu_{A,1}} + \frac{\partial(\eta w)}{\partial\nu_1}\right\|_{H^{\frac{1}{2}}(\Gamma_1')} \leq 2\sqrt{2}c(\eta)\left(\|\gamma_1 w_x\|_{H^{\frac{1}{2}}(\Gamma_1')} + \|\gamma_1 w_y\|_{H^{\frac{1}{2}}(\Gamma_1')}\right)$$

$$+ c_3\left(\|\gamma_1 w_x\|_{L^2(\Gamma_1')} + \|\gamma_1 w_y\|_{L^2(\Gamma_1')}\right),$$

for some $c_3 > 0$, where γ_1 denotes the trace operator. Since $\partial\Omega'$ is of class $C^{0,1}$, we may apply Theorem 3.5 and inequality (3.1) (with $r = 1$, $s = 7/4$, $t = 2$) to obtain

$$\left\| \frac{\partial(\eta w)}{\partial \nu_{A,1}} + \frac{\partial(\eta w)}{\partial \nu_1} \right\|_{H^{\frac{1}{2}}(\Gamma_1')} \leq c_4 \left((c(\eta) + \varepsilon) \|\eta w\|_{H^2(\Omega')} + \varepsilon^{-3} \|\eta w\|_{H^1(\Omega')} \right), \quad (2.10)$$

for some $c_4 > 0$ and any $\varepsilon > 0$.

Due to (2.5), we may choose the support of η so small that

$$c(\eta) \leq \frac{1}{4c_1(2 + c_4)}.$$

Together with the choice $\varepsilon = 1/(4c_1c_4)$, the estimates (2.8)-(2.10) yield that

$$\|\eta w\|_{H^2(\Omega')} \leq c_5 \left(\|A(\eta w)\|_{L^2(\Omega')} + \left\| \frac{\partial(\eta w)}{\partial \nu_{A,1}} \right\|_{H^{\frac{1}{2}}(\Gamma_1')} + \|\eta w\|_{H^1(\Omega')} \right), \quad (2.11)$$

for some $c_5 > 0$.

Now we consider $(1 - \eta)w$: We choose an open domain $\Omega'' \subset \Omega'$ with a $C^{1,1}$ boundary, such that the boundary of Ω'' coincides with the boundary of Ω' outside of the set $\{\eta = 1\}$. Then estimate (3.2) immediately implies that

$$\|(1 - \eta)w\|_{H^2(\Omega')} \leq c_6 \left(\|A(1 - \eta)w\|_{L^2(\Omega')} + \left\| \frac{\partial(1 - \eta)w}{\partial \nu_{A,1}} \right\|_{H^{\frac{1}{2}}(\Gamma_1')} + \|(1 - \eta)w\|_{H^1(\Omega')} \right), \quad (2.12)$$

for some $c_6 > 0$.

Noting that for any $\rho \in C^{1,1}(\overline{O})$

$$\|A(\rho w)\|_{L^2(\Omega')} \leq \|\rho\, Aw\|_{L^2(\Omega')} + c_7\|w\|_{H^1(\Omega')},$$

for some $c_7 > 0$ (depending on ρ) and that, due to Theorems 3.1 and 3.5,

$$\left\|\frac{\partial(\rho w)}{\partial \nu_{A,1}}\right\|_{H^{\frac{1}{2}}(\Gamma_1')} \leq \left\|\rho \frac{\partial w}{\partial \nu_{A,1}}\right\|_{H^{\frac{1}{2}}(\Gamma_1')} + c_8\|w\|_{H^1(\Omega')},$$

for some $c_8 > 0$ (depending on ρ), the desired estimate (2.6) follows from (2.11), (2.12), and Theorem 3.1.

Since $H^2(\Omega')$ is compactly embedded in $H^1(\Omega')$, the assertion on the operator T_A being a semi-Fredholm operator is equivalent to estimate (2.6) (see, e.g., [8, Satz 12.12]). □

Lemma 2.4. *Let* $\Omega' \subset \mathbb{R}^2$ *satisfy the conditions* (i) *and* (ii) *in Remark* 2.2 *and let* A *be a differential operator of the form* (2.4), *where the coefficients satisfy the conditions in* (2.5). *Moreover, we assume that the operator* T_A, *defined by* (2.7), *is one to one.*

Then it holds that T_A *maps the space* $H^2(\Omega')$ *onto* $L^2(\Omega') \times H^{\frac{1}{2}}(\Gamma_1')$ *if* $\omega_1 \in (0, \pi)$ *and onto a closed subspace of* $L^2(\Omega') \times H^{\frac{1}{2}}(\Gamma_1')$ *of codimension* 1 *if* $\omega_1 \in (\pi, 2\pi)$.

Proof. In a first step, we consider the special case $Aw = Lw := -\Delta w + \lambda w$, $\lambda > 0$. It is obvious that L satisfies the conditions in (2.5). Moreover, (2.3) implies that $\mathcal{N}(T_L) = \{0\}$. Note that $\frac{\partial w}{\partial \nu_{A,1}} = \frac{\partial w}{\partial \nu_1}$.

Let $f \in L^2(\Omega')$, $g_1 \in H^{\frac{1}{2}}(\Gamma_1')$, and let $w \in H^1(\Omega')$ be the unique weak solution to

$$Lw = f \quad \text{in } \Omega',$$

$$\frac{\partial w}{\partial \nu_1} = g_1 \quad \text{on } \Gamma_1'.$$

Moreover, let $O \subset \Omega'$, $\Omega'' \subset \Omega'$, and η be as in the proof of Lemma 2.3. Since

$$\Delta(\eta w) = -\eta f + 2\nabla\eta \cdot \nabla w + \Delta\eta w - \lambda\eta w \in L^2(\Omega'),$$

$$\frac{\partial(\eta w)}{\partial \nu_1} = \eta g_1 + \frac{\partial \eta}{\partial \nu_1}\gamma_1 w \in H^{\frac{1}{2}}(\Gamma_1'),$$

it immediately follows with Theorem 3.9 that

$$(1-\eta)w \in H^2(\Omega').$$

Furthermore, Theorem 3.10 implies that

$$\eta w \in H^2(\Omega') \text{ if } \omega_1 \in (0, \pi), \quad \eta w - c_1\rho_1 \in H^2(\Omega') \text{ if } \omega_1 \in (\pi, 2\pi),$$

for some $c_1 > 0$. This already implies the assertion in case $\omega_1 \in (0, \pi)$.

Let us now assume that $\omega_1 \in (\pi, 2\pi)$. Then $w - c_1\rho_1 \in H^2(\Omega')$. Since Lemma 2.3 applied to $A = L$ implies that $T_L(H^2(\Omega'))$ is closed in $L^2(\Omega') \times H^{\frac{1}{2}}(\Gamma_1')$, it follows that

$$(f, g_1) = T_L v + c_1 z, \quad v \in H^2(\Omega'), \quad z := (I-P)(L\rho_1, 0) \neq 0,$$

where P is the orthogonal projector onto $T_L(H^2(\Omega'))$. Thus, whenever

$$\langle (f, g_1), z\rangle_{L^2(\Omega')\times H^{\frac{1}{2}}(\Gamma_1')} = 0,$$

the solution $u \in H^2(\Omega')$. This proves the assertion.

Let us now consider the general case: We define the operators

$$T(t) := tT_A + (1-t)T_L, \quad t \in [0, 1].$$

Since the corresponding operators $tA + (1-t)L$ satisfy the assumptions of Lemma 2.3, we may conclude that $T(t)$ is a semi-Fredholm operator for all $t \in [0, 1]$. Noting that the operators $T(t)$ depend continuously on t, it follows with [4, Theorem IV-5.17] (see also the proof of [4, Theorem IV-5.22]) that the index of $T(t)$, i.e., the difference of the dimension of $\mathcal{N}(T(t))$ and the codimension of $\mathcal{R}(T(t))$, is constant. Since the operators T_L and T_A are one to one, $\mathcal{R}(T_L)$ and $\mathcal{R}(T_A)$ have the same codimension. The assertions now follow with the results for the special case $A = L$. □

Now, we are able to prove our main result for curvilinear polygons using the same notation as above.

Theorem 2.5. *Let* Ω *be an open bounded connected subset of* $\mathbb{R}^2$, *whose boundary is a curvilinear polygon of class* $C^{1,1}$ *with N segments* Γ_j, *corners* S_j, *and interior angles* $\omega_j \in (0, 2\pi)\setminus\{\pi\}$, $j = 1, \ldots, N$. *Then problem* (2.1) *has a unique solution* $u \in H^1(\Omega)$ *for all* $f \in L^2(\Omega)$ *and* $g_j \in H^{\frac{1}{2}}(\Gamma_j)$. *Moreover, there are unique numbers* c_j *such that*

$$u - \sum_{\substack{j=1 \\ \omega_j \in (\pi, 2\pi)}}^{N} c_j \rho_j \in H^2(\Omega),$$

where the functions ρ_j *are defined as in* (3.3). *In addition, it holds that*

$$u \in H^s(\Omega) \quad \textit{for all} \quad s \le 2 \quad \textit{with} \quad s < 1 + \inf\left\{\frac{\pi}{\omega_j} : 1 \le j \le N\right\}.$$

Proof. Let $f \in L^2(\Omega)$, $g_j \in H^{\frac{1}{2}}(\Gamma_j)$, and let $u \in H^1(\Omega)$ be the unique weak solution to (2.1). Let η_j be a C^∞ cut-off function equal to 1 near S_j such that the support does not intersect Γ_k for $k \neq j$ and $j + 1$. Then

$$L(\eta_j u) = \tilde{f}_j := \eta_j f - 2\nabla\eta_j \cdot \nabla u - \Delta\eta_j u \in L^2(\tilde{\Omega}_j),$$

$$\frac{\partial(\eta_j u)}{\partial \nu_j} = \widetilde{g}_j := \eta g_j + \frac{\partial \eta_j}{\partial \nu_j} \gamma_j u \in H^{\frac{1}{2}}(\widetilde{\Gamma}_j),$$

where $\widetilde{\Omega}_j \subset \Omega$ is an open domain, whose boundary is a curvilinear polygon of class $C^{1,1}$ with only one corner S_j that coincides with the boundary of Ω near S_j. Moreover, we assume that $\widetilde{\Omega}_j$ contains the support of η_j and $\widetilde{\Gamma}_j := \partial\widetilde{\Omega}_j \setminus S_j$.

Let ϕ_j be a transformation satisfying conditions (i)-(iv) in Remark 2.1 near S_j (after a rotation). Note that if $\operatorname{supp}(\eta_j)$ is small enough, one can choose $\widetilde{\Omega}_j$ so small that it is contained in W_j. Then $\Omega'_j := \psi(\widetilde{\Omega}_j) = \phi^{-1}(\widetilde{\Omega}_j)$ satisfies conditions (i) and (ii) in Remark 2.2. It was mentioned in this remark that $w_j := (\eta_j u) \circ \phi_j$ is such that

$$A_j w_j = \widetilde{f}_j \circ \phi_j \in L^2(\Omega'_j),$$

$$\frac{\partial w_j}{\partial \nu_{A,j}} = -(\widetilde{g}_j \circ \phi_j) \det D\phi_j \in H^{\frac{1}{2}}(\Gamma'_j), \quad \Gamma'_j := \partial\Omega'_j \setminus (0, 0),$$

where A_j is a differential operator of the form (2.4), where the coefficients satisfy the conditions in (2.5). Since (2.3), Theorem 3.1 (note that $\det D\phi_j \in C^{0,1}(\overline{\Omega'_j})$ is bounded away from 0), and Theorem 3.2 imply that T_{A_j}, defined as in (2.7), is one to one, we may apply Lemma 2.4 to conclude that $w_j \in H^2(\Omega'_j)$ if $\omega_j \in (0, \pi)$. Thus, due to Theorem 3.2 $\eta_j u \in H^2(\Omega)$ then.

Let us now consider the case $\omega_j \in (\pi, 2\pi)$: We will show that there is a unique number c_j such that

$$\eta_j u - c_j \rho_j \in H^2(\Omega). \tag{2.13}$$

It is an immediate consequence of Theorem 3.4 that

$$\rho_j \in H^s(\Omega) \quad \text{for all} \quad s < 1 + \frac{\pi}{\omega_j} < 2. \tag{2.14}$$

A direct calculation shows that

$$L\rho_j := -\Delta\rho_j + \lambda\rho_j \in L^2(\widetilde{\Omega}_j). \tag{2.15}$$

We will also show that

$$\frac{\partial\rho_j}{\partial\nu_j} \in H^{\frac{1}{2}}(\widetilde{\Gamma}_j). \tag{2.16}$$

Of course, it is only necessary to consider points close to S_j : Let us consider the part of $\widetilde{\Gamma}_j$ that coincides with a part of Γ_{j+1}. Then points on the boundary may be written as $(t, \varphi(t))$ with

$$\varphi \in C^{1,1}[0, a], \qquad \varphi(0) = 0 = \varphi'(0), \tag{2.17}$$

for some $a > 0$. Points on that part of the boundary of $\widetilde{\Gamma}_j$ that coincides with a part of Γ_j may be written as $t(\cos\omega_j, \sin\omega_j) + \varphi(t)(-\sin\omega_j, \cos\omega_j)$, where φ again satisfies the conditions in (2.17). In both cases, we obtain that

$$\frac{\partial\rho_j}{\partial\nu_j} = \lambda_j r_j^{\lambda_j - 1} \frac{\varphi'(t)\cos((\lambda_j - 1)\theta_j) + \sin((\lambda_j - 1)\theta_j)}{(1 + \varphi'(t)^2)^{\frac{1}{2}}},$$

$$\lambda_j := \frac{\pi}{\omega_j}, \quad r_j := (t^2 + \varphi(t)^2)^{\frac{1}{2}}, \quad \theta_j := \arctan\frac{\varphi(t)}{t}.$$

This together with (2.17) implies (2.16). Together with (2.15) and Theorems 3.1 and 3.2, it then follows that

$$T_{A_j}(\rho_j \circ \phi_j) \in L^2(\Omega'_j) \times H^{\frac{1}{2}}(\Gamma'_j).$$

Since Lemma 2.4 implies that $T_{A_j}(H^2(\Omega'_j))$ is a closed subspace in $L^2(\Omega'_j) \times H^{\frac{1}{2}}(\Gamma'_j)$ of codimension 1, we now obtain that

$$w_j - c_j(\rho_j \circ \phi_j) \in H^2(\Omega'_j),$$

where

$$c_j := \frac{\left\langle (\tilde{f}_j \circ \phi_j, -(\tilde{g}_j \circ \phi_j)\det D\phi_j), z_j \right\rangle_{L^2(\Omega'_j)\times H^{\frac{1}{2}}(\Gamma'_j)}}{\|z_j\|_{L^2(\Omega'_j)\times H^{\frac{1}{2}}(\Gamma'_j)}},$$

$$z_j := (I - P_j)T_{A_j}(\rho_j \circ \phi_j),$$

and P_j is the orthogonal projector onto $T_{A_j}(H^2(\Omega'_j))$. Another application of Theorems 3.1 and 3.2 now yields (2.13).

Using the same idea as in the proof of Lemma 2.4, Theorem 3.9 implies that

$$(1 - \sum_{j=1}^{N} \eta_j)u \in H^2(\Omega).$$

This together with (2.13) implies the first assertion whereas the second assertion immediately follows from (2.14). □

Remark 2.6. We may now apply this result to the reconstruction problem from adaptive optics mentioned in the Introduction. The regions Ω_l are unions of annuli. Thus, the boundary is a curvilinear polygon. According to Theorem 2.5, the iterates of the Landweber-Kaczmarz regularization for the three examples in Figure 1.1 are in the space H^s with

Case 1. $z = 16 \quad s < 1.629$,

Case 2. $z = 10 \quad s < 1.689$,

Case 3. $z = 4 \quad s < 1.886$,

where in Case 2, the infimum is attained at the interior corners.

Thus, the smoothness of the regularized solutions to the atmospheric tomography problem is rather close to the expected one (see the Introduction). This might be the reason why H^1-Kaczmarz reconstructors usually yield better approximations than just L^2-Kaczmarz reconstructors.

3. Some Properties of Sobolev Spaces

For the convenience of the readers, in this section, we collect some properties of Sobolev spaces that are needed in the sections above. Proofs can be found in [3, 5]. There the results are formulated for general Sobolev spaces $W^{s,p}$. We only need them for the spaces $W^{s,2} = H^s$.

Theorem 3.1. *Let* Ω *be an open bounded subset of* $\mathbb{R}^n$ *and let* $\rho \in C^{k,\alpha}(\overline{\Omega})$ *with* $k + \alpha \geq |s|$ *if s is an integer and* $k + \alpha > |s|$ *otherwise. Then* $\rho u \in H^s(\Omega)$ *for all* $u \in H^s(\Omega)$ *and there is a constant* $c = c(\rho, s)$ *such that*

$$\|\rho u\|_{H^s(\Omega)} \leq c\|u\|_{H^s(\Omega)}.$$

Theorem 3.2. *Let* Ω_1 *and* Ω_2 *be two open bounded subsets of* $\mathbb{R}^n$ *and let* ϕ *be a* $C^{k,1}$ *diffeomorphism between them:* $\phi : \Omega_1 \to \Omega_2$, $\phi^{-1} : \Omega_2 \to \Omega_1$. *Then the operators*

$$T_1 : H^s(\Omega_2) \to H^s(\Omega_1) \qquad T_2 : H^s(\Omega_1) \to H^s(\Omega_2)$$

$$u \mapsto u \circ \phi \qquad\qquad v \mapsto v \circ \phi^{-1}$$

are continuous for $s \leq k + 1$.

Theorem 3.3. *Let* $t > s > r \geq 0$ *and assume that* Ω *is an open bounded subset of* $\mathbb{R}^n$ *with a* $C^{0,1}$ *boundary. Then there exists a constant* $c = c(\Omega, t, s, r)$ *such that*

$$\|u\|_{H^s(\Omega)} \leq \varepsilon\|u\|_{H^t(\Omega)} + c\varepsilon^{-\frac{s-r}{t-s}}\|u\|_{H^r(\Omega)}, \tag{3.1}$$

for all $u \in H^t(\Omega)$ *and* $\varepsilon > 0$.

Theorem 3.4. *Let* Ω *be an open bounded connected subset of* $\mathbb{R}^2$, *whose boundary is a curvilinear polygon. Assume that* $0 \in \Gamma$ *and let* V *be a neighbourhood of* 0 *such that*

$$V \cap \overline{\Omega} \subset \{(r\cos\theta,\, r\sin\theta) : r \geq 0,\, a \leq \theta \leq b\},$$

with $b - a < 2\pi$. *Finally, let* u *be a function which is smooth in* $\overline{\Omega}\setminus\{0\}$ *and which coincides in polar coordinates with* $r^\alpha \varphi(\theta)$ *in* $V \cap \Omega$ *with* $\alpha \in \mathbb{C}$ *and* $\varphi \in C^\infty[a, b]$. *Then*

$$u \in H^s(\Omega\ ) \quad \textit{for} \qquad \operatorname{Re}\alpha > s - 1,$$

while

$$u \notin H^s(\Omega) \quad \textit{for} \qquad \operatorname{Re}\alpha \leq s - 1,$$

when $\operatorname{Re}\alpha$ *is not an integer.*

We also need some trace theorems.

Theorem 3.5. *Let* Ω *be an open bounded subset of* $\mathbb{R}^n$ *with a* $C^{k,1}$ *boundary* Γ. *Assume that* $s \leq k + 1$ *and that* $s - \frac{1}{2} = l + \sigma$ *with* $l \in \mathbb{N}_0$ *and* $0 < \sigma < 1$. *Then the mapping*

$$u \mapsto \left\{\gamma u,\, \gamma\frac{\partial u}{\partial \nu},\, \cdots,\, \gamma\frac{\partial^l u}{\partial \nu^l}\right\},$$

which is defined for $u \in C^{k,1}(\overline{\Omega})$, *has a unique continuous extension as an operator from* $H^s(\Omega)$ *onto* $\prod_{j=0}^{l} H^{s-j-\frac{1}{2}}(\Gamma)$. *Moreover, this operator has a linear continuous right inverse.*

The operator γ above denotes the trace operator, the higher order normal derivatives are defined by

$$\frac{\partial^l u}{\partial \nu^l} := \sum_{|\alpha|=l} \frac{l!}{\alpha!} D^\alpha u \nu^\alpha.$$

Using the notation as in the section above, we get the following result for curvilinear polygons:

Theorem 3.6. *Let Ω be an open bounded connected subset of $\mathbb{R}^2$, whose boundary is a curvilinear polygon of class $C^{k,1}$ and assume that $m \le k+1$. Then the mapping*

$$u \mapsto \left\{ \gamma_j \frac{\partial^l u}{\partial \nu_j^l} \right\}_{j=1, l=0}^{N, m-1},$$

is linear continuous from $H^m(\Omega)$ onto the subspace W of $\prod_{j=1}^{N} \prod_{l=0}^{m-1} H^{m-l-\frac{1}{2}}(\Gamma_j)$

defined by the following conditions:

Let L be any linear differential operator with $C^{0,1}$ coefficients and of order $d \le m-1$ and denote by $P_{j,l}$ the differential operator tangential to Γ_j such that $L = \sum_{l=0}^{d} P_{j,l} \frac{\partial^l}{\partial \nu^l}$. Then $\{f_{j,l}\} \in W$ satisfies the condition

$$\sum_{l=0}^{d} (P_{j,l} f_{j,l})(S_j) = \sum_{l=0}^{d} (P_{j+1,l}) f_{j+1,l}(S_j),$$

for $d < m-1$ and

$$\int_0^{\delta_j} \left| \sum_{l=0}^{d} (P_{j,l} f_{j,l})(x_j(-\sigma)) - \sum_{l=0}^{d} (P_{j+1,l} f_{j+1,l})(x_j(\sigma)) \right|^2 \frac{d\sigma}{\sigma} < \infty,$$

for $d = m-1$ and $\delta_j > 0$ sufficiently small.

Moreover, this mapping has a linear continuous right inverse.

Proof. The proof for a curvilinear polygon of class C^∞ and C^∞ coefficients of L was given in [3, Theorem 1.5.2.8]. An inspection of the proof shows that the result remains valid also for curvilinear polygons of class $C^{k,1}$ and $C^{0,1}$ coefficients of L, provided that $k \geq m-1$. The assertion about the continuous right inverse follows from the construction of the inverse in the proof of Theorem 1.5.2.4 together with Theorem 1.4.3.1 and Theorem 1.5.1.1 in [3] and Theorem 3.5. □

Corollary 3.7. *Let* Ω *be an open bounded connected subset of* $\mathbb{R}^2$, *whose boundary is a curvilinear polygon of class* $C^{1,1}$. *Then the mapping*

$$u \mapsto \left\{\gamma_j u,\ \gamma_j \frac{\partial u}{\partial \nu_j}\right\}_{j=1}^{N},$$

is linear continuous from $H^2(\Omega)$ *onto the subspace* W *of* $\prod_{j=1}^{N} H^{\frac{3}{2}}(\Gamma_j)\cdot H^{\frac{1}{2}}(\Gamma_j)$

defined by the following conditions: $\{f_{j,0},\ f_{j,1}\}_{j=1}^{N} \in W$ *is such that*

$$f_{j,0}(S_j) = f_{j+1,0}(S_j),$$

$$\int_0^{\delta_j} \left|\alpha_j f'_{j+1,0}(x_j(\sigma)) + f_{j,1}(x_j(-\sigma)) - \beta_j f_{j+1,1}(x_j(\sigma))\right|^2 \frac{d\sigma}{\sigma} < \infty,$$

and

$$\int_0^{\delta_j} \left|\alpha_j f'_{j,0}(x_j(-\sigma)) + \beta_j f_{j,1}(x_j(-\sigma)) - f_{j+1,1}(x_j(\sigma))\right|^2 \frac{d\sigma}{\sigma} < \infty,$$

for some $\delta_j > 0$ *sufficiently small, where*

$$\alpha_j := (\nu_j^2\nu_{j+1}^1 - \nu_j^1\nu_{j+1}^2)(S_j) \neq 0 \quad \textit{and} \quad \beta_j := (\nu_j^1\nu_{j+1}^1 + \nu_j^2\nu_{j+1}^2)(S_j)$$

(assuming that S_j *is the endpoint of* $\overline{G_j}$ *and starting point of* $\overline{G_{j+1}}$*). Here,* $f'_{j,0}$ *and* $f'_{j+1,0}$ *denote the tangential derivatives* $\frac{\partial}{\partial\tau_j} f_{j,0}$ *and* $\frac{\partial}{\partial\tau_{j+1}} f_{j+1,0}$*, respectively.*

Moreover, this mapping has a linear continuous right inverse.

Proof. The conditions can be directly derived from the ones in Theorem 3.6 noting that for the differential operator $Lu := a_{00}u + a_{01} D_1u + a_{10}D_2u$ the differential operators $P_{j,l}$ are given by

$$P_{j,0}u = a_{00}u + (a_{01}\nu_j^2 - a_{10}\nu_j^1)\frac{\partial u}{\partial\tau_j},$$

$$P_{j,1}w = (a_{01}\nu_j^1 + a_{10}\nu_j^2)w.$$

□

Remark 3.8. It is an immediate consequence of this corollary that for any functions $g_j \in H^{\frac{1}{2}}(\Omega)$, $j = 1, \ldots, N$, there exists a function $u \in H^2(\Omega)$ such that $\frac{\partial u}{\partial\nu_j} = g_j$ satisfying the estimate

$$\|u\|_{H^2(\Omega)} \le c\sum_{j=1}^{N}\|g_j\|_{H^{\frac{1}{2}}(\Gamma_j)},$$

for some $c > 0$. This follows by setting $f_{j,1} := g_j$ and choosing $f_{j,0}$ such that the conditions in Corollary 3.7 are satisfied and that

$$\sum_{j=1}^{N}\|f_{j,0}\|_{H^{\frac{3}{2}}(\Gamma_j)} \le \bar{c}\sum_{j=1}^{N}\|g_j\|_{H^{\frac{1}{2}}(\Gamma_j)},$$

for some $\bar{c} > 0$, which is trivially always possible.

Finally, we cite some results about solutions to boundary value problems starting with smooth domains (cf. [3, Corollary 2.2.2.6]).

Theorem 3.9. *Let* Ω *be a bounded open subset of* $\mathbb{R}^2$ *with* $C^{1,1}$ *boundary* Γ *and let* A *be a strongly elliptic differential operator of the form* (2.4) *with coefficients* $a_{ij} \in C^{0,1}(\overline{\Omega})$, $a_1 = a_2 = 0$, *and* $\lambda > 0$. *Then the mapping*

$$u \mapsto \left\{Au, \frac{\partial u}{\partial \nu_A}\right\},$$

is invertible from $H^2(\Omega)$ *onto* $L^2(\Omega) \times H^{\frac{1}{2}}(\Gamma)$. *Thus,*

$$\|u\|_{H^2(\Omega)} \leq c\left(\|Au\|_{L^2(\Omega)} + \left\|\frac{\partial u}{\partial \nu_A}\right\|_{H^{\frac{1}{2}}(\Gamma)}\right), \tag{3.2}$$

for some $c > 0$.

Theorem 3.10. *Let* Ω *be a rectilinear polygon with N segments* Γ_j, *corners* S_j, *and interior angles* $\omega_j \in (0, 2\pi)\setminus\{\pi\}$, $j = 1, \dots, N$. *Then, there exists a solution* $u \in H^1(\Omega)$, *unique up to an additive constant, of the Neumann problem*

$$\Delta u = f \quad \text{in } \Omega,$$

$$\frac{\partial u}{\partial \nu_j} = g_j \quad \text{on } \partial\Gamma_j, \quad j = 1, \dots, N,$$

for $f \in L^2(\Omega)$ *and* $g_j \in H^{\frac{1}{2}}(\Gamma_j)$ *if and only if*

$$\langle f, 1\rangle_{L^2(\Omega)} - \sum_{i=1}^{N}\langle g_j, 1\rangle_{L^2(\Gamma_j)} = 0.$$

Moreover, there are unique numbers c_j *such that*

$$u - \sum_{\substack{j=1 \\ \omega_j \in (\pi, 2\pi)}}^{N} c_j \rho_j \in H^2(\Omega),$$

where the functions ρ_j *are defined as*

$$\rho_j(x, y) := r_j^{\frac{\pi}{\omega_j}} \cos\left(\frac{\pi}{\omega_j}\theta_j\right)\eta_j(x, y), \quad (x, y) \in \Omega, \tag{3.3}$$

(r_j, θ_j) *denote the polar coordinates of* (x, y) *with origin at* S_j *such that*

$\operatorname{supp}(\eta_j) \cap \Omega = S_j + \{(r_j \cos(\overline{\theta}_j + \theta_j), r_j \sin(\overline{\theta}_j + \theta_j)) : r_j \in (0, \varepsilon_j), \theta_j \in (0, \omega_j)\},$ *for some* $\varepsilon_j > 0$ *and* $\overline{\theta}_j \in [0, 2\pi)$. *The functions* η_j *are* C^∞ *cut-off functions only depending on* r_j *with value* 1 *in a neighbourhood of* S_j *and such that the support does not intersect* Γ_k *for* $k \neq j$ *and* $j+1$. *Note that*

$$0 \neq \Delta\rho_j \in C^\infty(\Omega), \qquad \frac{\partial \rho_j}{\partial \nu_k} = 0, \quad k = 1, \dots, N,$$

and that $\rho_j \in H^1(\Omega) \setminus H^2(\Omega)$ *if* $\omega_j \in (\pi, 2\pi)$.

In addition, there exists a constant $c > 0$ *such that*

$$\|u\|_{H^2(\Omega)} \leq c\left(\|\Delta u\|_{L^2(\Omega)} + \sum_{j=1}^{N} \left\|\frac{\partial u}{\partial \nu_j}\right\|_{H^{\frac{1}{2}}(\Gamma_j)}\right), \tag{3.4}$$

for all $u \in H^2(\Omega)$.

Proof. The proof immediately follows from Corollary 4.4.3.8, Theorem 4.3.1.4, and Theorem 1.5.2.8 in [3]. □

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